Collections of Sets, Counting

Discrete Structures (CS 173)

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Example: Image segmentation

- An image is divided into regions based on color or other appearance features
- Each region is a **set** of pixels
- The segmentation is a **collection** of regions, or a collection of sets of pixels

Sets of sets

The **powerset** of set S is the set of all possible subsets of S

Example: the powerset of $\{a, b, d\}$ is
$\{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$

Example: the powerset of the set of pixels is the set of all possible image regions (including discontinuous regions)

If we have n pixels, how many possible regions are there?

Functions on sets

Powersets may be used to specify the domain of a function that operates on sets.

Example 1: function to compute the mean intensity of pixels in a region

$$regionAve: \boldsymbol{P}(\{s_1, s_2, \ldots s_{npixels}\}), R^{npixels} \rightarrow R,$$

$$regionAve(S, \text{image}) = \frac{1}{|S|} \sum_{i \in S} \text{image}(i)$$

S is a set of pixel indices
i is an element of S (an image index)
$\text{image}(i)$ is the i^{th} pixel intensity

Functions on sets

Powersets may be used to specify the co-domain of a function that operates on sets

Example 2: function to return indices of red pixels

$$getRedIdx: R^{3n_{pixels}} \longrightarrow \boldsymbol{P}\left(\{1,2,..n_{pixels}\}\right),$$

$$getRedIdx(\text{red}, \text{green}, \text{blue}) =$$
$$\{i \in \left\{1..n_{pixels}\right\}: \ (\text{red}(i) > \text{blue}(i) + 0.2) \cap (\text{red}(i) > \text{green}(i) + 0.2)\}$$

i is an element of S (an image index)
$\text{red}(i)$ is the i^{th} intensity in red channel

Partitions

There are 10 kinds of people: those who understand binary and those who don't.

A **partition** of A is a collection of non-empty subsets of A, such that each element of A is contained by exactly one subset

Partitions with Finite Sets

A collection of sets $\boldsymbol{C}$ is a partition of A

(1) $\boldsymbol{C}$ covers A: $C_1 \cup C_2 \cup \cdots \cup C_n = A$

(2) No overlap within elements of $\boldsymbol{C}$: $C_i \cap C_j = \emptyset$ if $i \neq j$

(3) No element of $\boldsymbol{C}$ is empty

The segmentation is a **partition** of the image pixels: the regions do not overlap, are non-empty, and their union includes all pixels

Partitions with Infinite Sets

A collection of sets $\mathscr{C}$ is a partition of A if $\mathscr{C}$

1. covers all of A: $\bigcup_{X \in \mathscr{C}} X = A$
2. non-empty: $X \neq \emptyset$ for all $X \in \mathscr{C}$
3. no overlap: $X \cap Y = \emptyset$ for all $X, Y \in \mathscr{C}$, $X \neq Y$

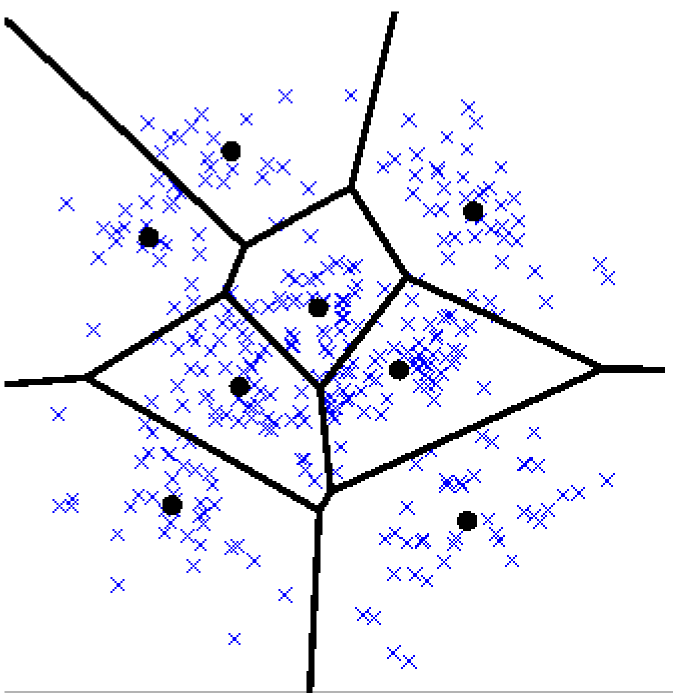
Example 1: The set of natural numbers can be partitioned into those between 0 and 9, 10 and 99, 100 and 999, 10^k and 10^{k+1}-1, etc. (elements of each set have the same number of digits)

Example 2: Any elements of the set of real numbers that round to the same integer are in a set. The collection of these sets is a partition on the set of real numbers.

Partitions of continuous spaces

Quantization: map R^n to a partition of R^n

- Clustering
- Decision trees



Represent any points that fall into each cell with an integer

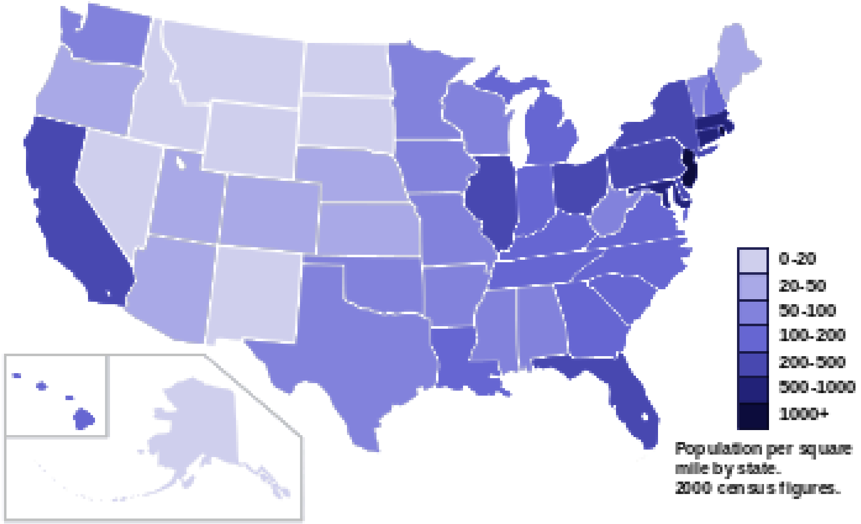
Treat all points within a cell as the same

Represent points in cell with summary statistics, such as mean

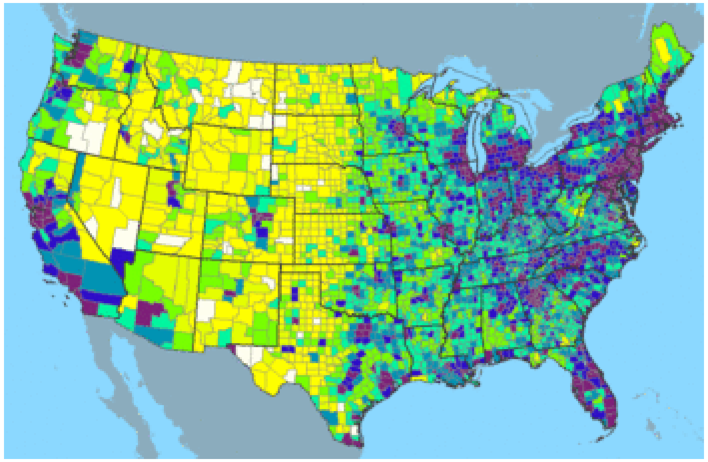
Simplifies matching and function fitting

Partitions of continuous spaces (R^2)

Population density by state



Population density by county



Population distribution over an area approximated by population within each state or county

Source: http://en.wikipedia.org/wiki/List_of_U.S._states_by_population_density

Partitions

Partition of the set $\{1,2,3\}$ *?*

$\{\{1,2\},\{3\}\}$

$\{\{1,2\},\{2,3\}\}$

$\{\{1\},\{2,3\},\emptyset\}$

Partition of Natural Numbers?

$\{\{\text{natural } x: \ x < 5\}, \{\text{natural } x: x > 5\}\}$

$\{\{\text{natural } x: \ x < 5\}, \{\text{natural } x: x \geq 5\}\}$

$\{\{\text{natural } x: \ x \leq 5\}, \{\text{natural } x: x \geq 5\}\}$

Partitions and equivalence sets

Define R on integers such that

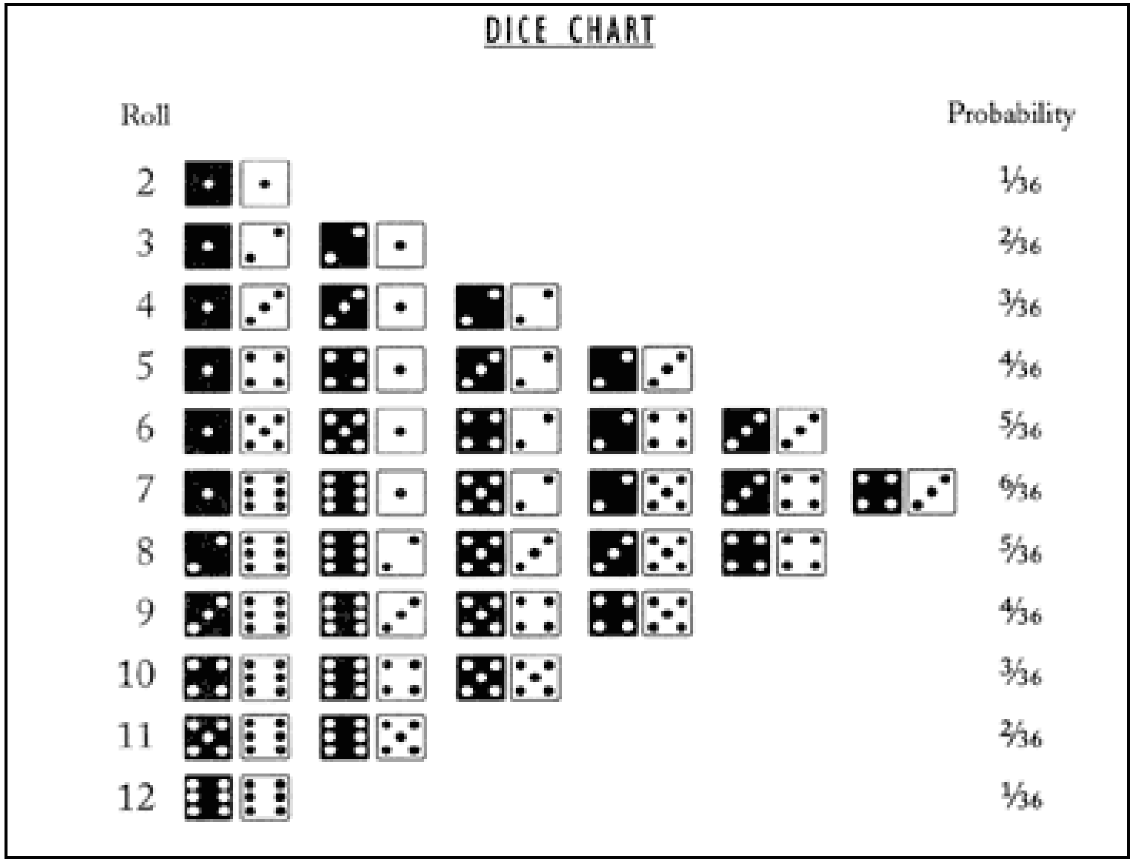
aRb if there is an $n \in \mathbb{Z}$ such that $2a + 5b = 3n$

Is R an equivalence relation?

Some nit-picky things to remember

- Important to distinguish between a one-element set and an element: $\{3\} \neq 3$
  - Common programming error (e.g., function is expecting a set but gets a number)
- Every powerset contains the empty set as an element
- An element of a collection (or set of sets) is a set

Permutations and Combinations



HW 9

2. **[12 points]**

There were 103 people at a bottle cap traders convention. Every trade involves two people. Show that at least one of these must be true: (1) some trades are profitable for only one person; (2) at least two people made profitable trades with the same number of other people. Note that someone could make 0 trades, so a person could trade with between 0 and 102 other people.

Counting

- How many permutations and combinations can I create by drawing from a bag of elements k times?
  - Does order of draw matter? (Permutation verses Combination)
  - Are elements from the bag replaced? (with repetition).
  - Can an element of the same type be chosen more than once?
- Foundation of computing probabilities of discrete events

- Questions
  - How many unique combinations of 3 toppings can I create if there are 8 kinds of toppings?
  - How many unique bridge/poker hands are possible?
  - If I flip a coin ten times, what is the chance that heads will come up exactly three times?
  - If I am trying to roll double ones and get to re-roll, what is the chance I will get it?

Permutations with Repetition

We use the term *permutations* when the order matters.

Choose k elements from n unique types with full replacement, order matters:

$$n^k \text{ permutations}$$

Examples:

- How many different (valid or invalid) 3-colorings are there for a graph with 15 nodes?
- How many unique symbols can I represent with 12 bits of data?

Permutations

Choose k elements from n unique types *with no replacement*, order matters:

$$\frac{\mathrm{n!}}{(n-k)!} \text{ permutations}$$

Examples:

- If I have 5 skittles of different flavors, how many different ways can I eat three of them (eating one at a time)? When you eat one, it is gone.
- Coins in your pocket (5,5,5,25,10). Once you use one, that one is gone.

Combinations

We use the term *combination* when order doesn't matter.

Choose k elements from n unique types with no replacement, order doesn't matter

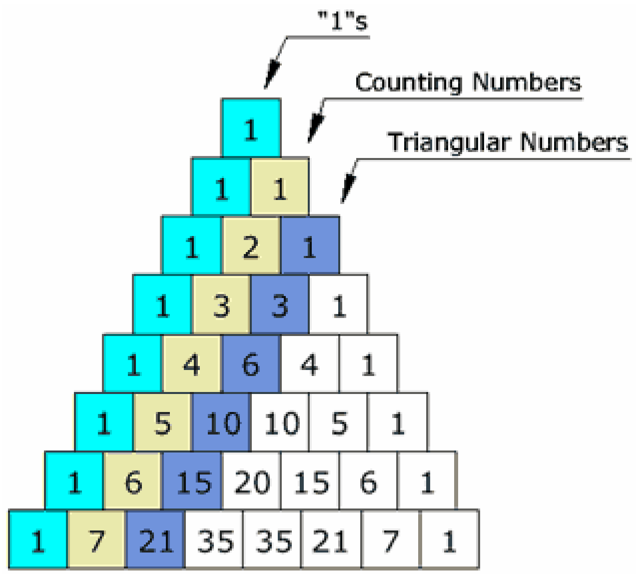
$$\frac{\mathrm{n}!}{k!(n-k)!} \equiv \binom{n}{k} \text{ combinations}$$

Examples:

- If I have 5 skittles of different flavors, how many different flavor combinations can I make by eating three at once?
- How many 3-topping pizzas can I create if there are 8 types of toppings?
- How many possible bridge hands can I have?
- How many possible bridge hands can two players have?

Pascal’s Triangle

“people are generally better persuaded by the reasons which they have themselves discovered than by those which have come into the mind of others”
--Blaise Pascal (1623-62)



To get n choose r combinations:
Go down n rows and walk down r boxes

http://www.mathsisfun.com/pascals-triangle.html

Pascal's triangle and Combinations

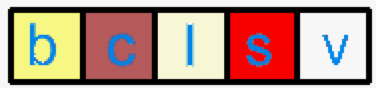
Tosses	Possible Results (Grouped)	Pascal's Triangle
1	H T	1, 1
2	HH HT TH TT	1, 2, 1
3	HHH HHT, HTH, THH HTT, THT, TTH TTT	1, 3, 3, 1
4	HHHH HHHT, HHTH, HTHH, THHH HHTT, HTHT, HTTH, THHT, THTH, TTHH HTTT, THTT, TTHT, TTTH TTTT	1, 4, 6, 4, 1
	... etc ...	

Combinations with Repetition

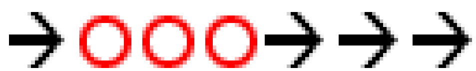
Let us say there are five flavors of vegan icecream:

banana, chocolate, lemon, strawberry and vanilla.

We can have three scoops. How many variations will there be?



Think about the ice cream being in boxes, we could say "move past the first box, then take 3 scoops, then move along 3 more boxes to the end" and we will have 3 scoops of chocolate!



How many positions are there and how many do we want to choose?

See: https://www.mathsisfun.com/combinatorics/combinations-permutations.html

Combinations with Repetition

Choose k elements from n unique types with replacement, order doesn't matter:

$$\frac{(k+n-1)!}{k!\,(n-1)!} \equiv \binom{k+n-1}{n-1} \equiv \binom{k+n-1}{k} \text{ combinations}$$

Examples

- If you flip a coin 10 times, how many unique head counts can you have?
- If you roll ten six-sided die, how many unique combos are possible?
- How many combos of 3 pizza toppings, with 8 options, can you make if you can choose the same topping multiple times?

Dice games (short break)

When rolling five dice at once, which is more likely, three-of-a-kind or a big-straight (2-3-4-5-6)?

Permutation of n objects of k-types

- Choosing an ordered set of k choices from a set of n objects
- Example: if you have Scrabble pieces

C, O, L, L, E, G, E

How many distinct permutations are there?

7! Includes permutations that are not distinct.

Which of the two L’s or E’s is used to make college is irrelevant. 2! ways to permute L’s and 2! ways to permute E’s. So the number of distinct orderings:

$$\frac{7!}{2!\,2!}$$

Permutation of n objects of k-types

- If we have n objects of k types, where n_1 are of first type, n_2 are of second type, …, n_k are of k^{th} type, the number of distinct permutations are:

$$\frac{n!}{n_1!\, n_2! \ldots n_{k!}}$$

Binomial theorem

Suppose you flip a coin n times. How many ways could you get k heads?

If the coin has an equal chance of being heads or tails, what is the chance of k heads?

What is the chance of there being from $0 \dots n$ heads?

What if there is a 60% chance of heads:

Chance of k heads?

Chance of $0 \dots n$ heads?

In general: $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

(binomial theorem)

Permutation problems

Examples:

- Suppose a slot machine has 6 dials which can each be set to {bell, cherry, 777, blank1, blank2}
  - How many possible permutations are there?
    - There are 5 choices for the first dial, for each of those there are 5 choices for the second, .. Thus: 5^6
  - How many permutations are there to with exactly three cherries?
    - If exactly 3 dials have cherries, the other 3 dials can have any one of the other 4 possibilities: 4^3. How many ways can you get 3 of 6 dials with cherries?
  - How many permutations are there to get at least three cherries?
    - Add the ways you can get 3, 4, 5 or 6 cherries.

Binomial Theorem Idea

Number of ways to choose *k* elements from an *n*-element set.

Write: $(x + y)^n$ as a product

Use the distributive law: each term of the form $x^{n-2}y^2$ each represents exactly one way of choosing 2 y's. Combining them, the coefficient represents all the ways of choosing 2 y's from the product.

See: https://en.wikipedia.org/wiki/Binomial_theorem#Combinatorial_interpretation

Binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Example: $(x+y)^3 =$

Next week

- Finite state machines